Some Results on a Fiber-Reinforced Viscoelastic Beam

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Introduction

IN the past several years, considerable use has been made of fiber-reinforced viscoelastic materials in the fabrication of structures. Many of these materials consist of long, thin fibers imbedded in a viscoelastic matrix. Most analyses of structures composed of these materials, model them as either anisotropic viscoelastic or elastic materials. These models, however, ignore an inherent property of this class of materials: that the behavior in tension is different than that in compression. This results from the tendency of the fibers to buckle when they are under compression.

When the material is either all in compression, or all in tension, this discontinuity in the modulus presents no difficulty. When the material is part in tension and part in compression, however, it complicates the analysis immeasurably. This effect is magnified by the viscoelastic nature of the material, because the regions of tension and compression change with time.

The first investigation of the behavior of this type material was made by Nachlinger and Leininger.[1] They investigated the stresses in a beam under pure bending. Unfortunately, for the case of a constant applied moment, they only obtained numerical results. In this Note, I will partially eliminate this lack of information. I will again consider a beam of rectangular cross section subjected to a constant moment. Instead of numerical results, however, I will present explicit forms for the asymptotic values of the stress, curvature, and position of the neutral axis.

Development of the Equations

The stress-strain relation that we will adopt can be written as

$$\sigma(t) = [\alpha + E(o)]\epsilon(t) + \int_o^t \dot{E}(\tau)\epsilon(t-\tau)d\tau \qquad (1)$$

where $E(t)$ is the relaxation modulus and α takes into account the strength due to the fibers. For reasons given above, we will take α to be a discontinuous function of ϵ given by

$$\alpha = \begin{cases} \alpha_t & \epsilon > 0 \\ \alpha_c & \epsilon < 0 \end{cases} \qquad (2)$$

The relevant balance equations are

$$\int_A \sigma dA = 0$$

and

$$\int_A \sigma y dA = -M \qquad (3)$$

where y is measured from the neutral axis. The usual strain-curvature assumption will also be made, i.e.,

$$\epsilon = -y/\rho \qquad (4)$$

where ρ is the radius of curvature.

If the beam is now taken to be of rectangular cross section with a depth of b and a height of h, these equations can be combined to yield the two governing equations:

$$\int_{-c}^0 \alpha_t y(t)dy + \int_o^{h-c} \alpha_c y(t)dy + \int_{-c}^{h-c}\left[E(o)y(t) + \int_0^t \frac{\rho(t)}{\rho(t-\tau)}\{y(t-\tau)\dot{E}(\tau)d\tau\}\right]dy = 0 \qquad (5)$$

and

$$\int_{-c}^0 \alpha_t y^2(t)dy + \int_o^{h-c} \alpha_c y^2(t)dy + \int_{-c}^{h-c}\left[E(o)y^2(t) + \int_o^t \frac{\rho(t)}{\rho(t-\tau)} y^2(t-\tau)\dot{E}(\tau)d\tau\right]dy = \frac{M\rho(t)}{b}\dagger$$

where c is the distance from the bottom of the beam to the neutral axis, and a function of time.

After the spatial integration is carried out, Eqs. (5) become:

$$\beta^2(t) + \gamma[1 - 2\beta(t)] + \rho(t)\int_0^t \frac{\dot{E}(\tau)}{d}\gamma(t-\tau)d\tau = 0$$

and (6)

$$\beta^3(t) + \gamma[1 - 3\beta(t) + 3\beta^2(t)] + \frac{\rho(t)}{d}\int_0^t \dot{E}(\tau)\alpha(t-\tau)d\tau = A\rho(t)$$

where

$$A = 3M/bdh^3, \quad d = \alpha_c - \alpha_t, \quad \beta(t) = c(t)/h$$

$$\gamma = [\alpha_c + E(o)]/d$$

$$\gamma(t-\tau) = [1 - 2\beta(t-\tau)]/\rho(t-\tau)$$

$$\alpha(t-\tau) = [1 - 3\beta(t-\tau) + 3\beta^2(t-\tau)]/\rho(t-\tau)$$

If these two equations are solved for β and ρ, Eqs. (1, 2, and 4) can then be used to compute the stresses.

Results

Since Eqs. (6) are quite difficult to solve, we do not have a simple formula for the stress as a function of time. In this section, we will, however, develop simple formulas for the limit of the stress and curvature as time increases without bound. If, as was indicated in the results of Nachlinger and Leininger, the stress is monotone increasing, then these results will also be the least upper bound on the stress.

If we formally take the limit of Eqs. (6) as $t \to \infty$, and denote

$$\lim_{t\to\infty} g(t) \text{ as } g^*$$

we immediately obtain:

$$\beta^{2*} + \gamma[1 - 2\beta^*] + \frac{\rho^*}{d}\lim_{t\to\infty}\int_0^t \dot{E}(\tau)\gamma(t-\tau)d\tau = 0$$

and (7)

$$\beta^{3*} + \gamma[1 - 3\beta^* + 3\beta^{*2}] + \frac{\rho^*}{d}\lim_{t\to\infty}\int_0^t \dot{E}(\tau)\alpha(t-\tau)d\tau = A\rho^*$$

Before β^* and ρ^* can be obtained, we must first evaluate the limits of the two integrals. In the Appendix, we will establish that

$$\lim_{t\to\infty}\int_0^t \dot{E}(\tau)\xi(t-\tau)d\tau = [E^* - E(o)]\xi^* \qquad (8)$$

if

$$\lim_{t\to\infty}\xi(t) = \xi^*, \lim_{t\to\infty}E(t) = E^*$$

ξ and $\dot{E}$ are continuous on $[o,\infty)$, and $\dot{E}(t) < 0$. Using this result, Eqs. (7) become

$$\beta^{*2} + [(\alpha_c + E^*)/d][1 - 2\beta^*] = 0 \qquad (9)$$

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† It should be noted that these equations are slightly different than those of Nachlinger and Leininger. While this difference does not effect the limiting values, it does change the rate at which things decay.

and

$$\beta^{*3} + [(\alpha_c + E^*)/d][1 - 3\beta^* + 3\beta^{*2}] = A\rho^*$$

or solving for β^* and ρ^*, we finally obtain

$$\beta^* = [(\alpha_c + E^*)/d][1 - \{d/(1 - \alpha_c + E^*)\}^{1/2}]$$

$$\rho^* = h^3 b\left(\frac{\alpha_c + E^*}{3M}\right)\left\{1 + 5\left(\frac{\alpha_c + E^*}{d}\right) + \left[10\left(\frac{\alpha_c + E^*}{d}\right) - 4\right]\beta^*\right\} \tag{10}$$

Since we now have β^* and ρ^*, we can calculate the stresses. If we denote $\sigma_t{}^*$ as the maximum tensile stress and $\sigma_c{}^*$ as the maximum compressive stress, we get, by using our result (8) along with Eqs. (1, 2, and 4),

$$\sigma_t{}^* = [\alpha_t + E^*]\beta^* h/\rho^*$$

and

$$\sigma_c{}^* = [\alpha_c + E^*](1 - \beta^*)h/\rho^* \tag{11}$$

Conclusions

Although the results presented are for the simplest case, that of a rectangular beam, it is obvious that they could easily be extended to beams of arbitrary cross section, by using result (8). The only complications would be algebraic. The real importance of these results lies in their simplicity. They enable one to obtain the asymptotic values of the stress by simple calculations instead of the solution of two nonlinear integral equations. It should also be noticed that the stresses given by these results are, in general, significantly different than those obtained if the difference in moduli is ignored.

Appendix

We will prove the following:

Theorem

If

$$\lim_{t\to\infty}\xi(t) = \xi^*, \quad \lim_{t\to\infty}E(t) = E^*$$

and if ξ and $\dot{E}$ are continuous on $[o,\infty)$, and if $\dot{E}(t) < 0$, then

$$\lim_{t\to\infty}\int_o^t \xi(t-\tau)\dot{E}(\tau)d\tau = \xi^*[E^* - E(o)]$$

Proof

We know from the hypothesis that given $\epsilon_1 > o$, then there is a T_1, such that

$$|\xi(t) - \xi^*| < \epsilon, \quad \text{whenever} \quad t > T_1$$

Also if $\epsilon_2 > o$, then there is a T_2, such that

$$|E(t) - E^*| < \epsilon_2 \quad \text{whenever} \quad t > T_2$$

We can now write

$$\int_o^t \xi(t-\tau)\dot{E}(\tau)d\tau = \int_o^{t^*} \xi(t-\tau)\dot{E}(\tau)d\tau + \int_{t^*}^t \xi(t-\tau)\dot{E}(\tau)d\tau$$

or if we use the mean-value theorem for integrals

$$\int_o^t \xi(t-\tau)\dot{E}(\tau)d\tau = \xi(t-\alpha)[E(t^*) - E(o)] + \xi(t-\beta)[E(t) - E(t^*)]$$

where $\alpha\epsilon[o,t^*]$, $\beta\epsilon[t^*,t]$. After some manipulation the preceding equation can be written as

$$\left|\int_o^t \xi(t-\tau)\dot{E}(\tau)d\tau - \xi^*[E^* - E(o)]\right| = |[\xi(t-\alpha) - \xi^*]E(o) - \xi^*[E^* - E(t^*)] + [\xi(t-\alpha) - \xi^*]E(t^*) + \xi(t-\beta)[E(t^*) - E(t)]|$$

or

$$\left|\int_o^t \xi(t-\tau)\dot{E}(\tau)d\tau - \xi^*[E^* - E(o)]\right| \le |\xi(t-\alpha) - \xi^*||E(o)| + |\xi^*||E^* - E(t^*)| + |\xi(t-\alpha) - \xi^*||E(t^*)| + |\xi(t-\beta)|[|E(t^*) - E^*| + |E(t) - E^*|]$$

where as before $\alpha\epsilon[o,t^*]$. Now if $t - \alpha > T_1$ then $|\xi(t - \alpha) - \xi^*| < \epsilon_1$
and,

$$\text{if } t^* > T_2, \quad \text{then} \quad |E^*E(t^*)| < \epsilon_2$$

and if

$$t > T_2, \quad \text{then} \quad |E(t)E^*| < \epsilon_2$$

Now if we pick $t > T_1 + t^* > T_1 + T_2$, all the aforementioned conditions are satisfied and

$$\left|\int_o^t \xi(t\tau)\dot{E}(\tau)d\tau - \xi^*[E^* - E(o)]\right| < M\epsilon_1 + N\epsilon_2 < \epsilon^*$$

whenever $t > T_1 + T_2$. Which is the desired result.

Reference

[1] Nachlinger, R. R. and Leininger, J. R., "Bending of a Beam Made of a Fiber-Reinforced Viscoelastic Material," *AIAA Journal*, Vol. 7, No. 10, Oct. 1969, pp. 2016–2017.